$$\text{curl } (\vec{v}) = \nabla \times \vec{v} \qquad\qquad \text{div } (\vec{v}) = \nabla \cdot \vec{v}$$

$$\vec{v} = (P, Q, R)$$

prop: ① $\text{curl }(\nabla f) = 0$ and ② $\text{div }(\text{curl}(\vec{v})) = 0$.

NOTE ① the divergence of a vector field calculates

"how badly does the v.f want to leave a bounded set."

② the curl itself is a measure of "how swirly" a v.f

wants to be...

$\rightarrow$ the curl itself is "swirly" thing.

Recasting Green's theorem

Let $\vec{v} = \langle P, Q, 0 \rangle$ have cts partial derivatives on an open region R

containing D, when D is a closed region w/ a piecewise-smooth boundary curve.

Then $\iint_D \text{curl}(\vec{v}) \cdot \vec{k} \, dA = \int_{\partial D} \vec{v} \cdot d\vec{r}$ and

$\int_{\partial D} \vec{v} \cdot (y'(t)\vec{i} - x'(t)\vec{j}) \cdot \dfrac{1}{|r(t)|} \, ds = \iint_D \text{div}(\vec{v}) \, dA$

$\text{curl}(\vec{v}) = \det \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = (-Q_z, +P_z, Q_x - P_y)$

$\text{curl}(\vec{v}) \cdot k = Q_x - P_y = \dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}$

$\downarrow$

$(0, 0, 1)$

So first equality is $\iint_D \text{curl}(\vec{v}) \cdot \vec{k} \, dA = \iint_D \left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} \right) dA \overset{\text{Green's theorem}}{=} \int_{\partial D} \vec{v} \cdot d\vec{r}$

So firm equality is $\iint_D \text{curl}(\vec{v}) \cdot \vec{k} \, dA = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_{\partial D} \vec{v} \, d\vec{r}$

$$\text{div}(\vec{v}) = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot (P, Q, 0)$$

$$= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$$

$\iint_D \text{div}(\vec{v}) \, dA = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA$ ⟶ v.f $\vec{w} = (-Q, P, 0)$

$= \int_{\partial D} \vec{w} \cdot d\vec{r}$

Green's theorem

$= \int_{t=a}^{b} (-Qx'(t) + Py'(t)) \, dt$

$= \int_{t=a}^{b} \langle P, Q \rangle \cdot \langle y', x' \rangle \, dt$

$= \int_{\partial D} \vec{v} \cdot (y'(t)\vec{i} - x'(t)\vec{j}) \frac{1}{|r'(t)|} \, ds.$ / Stoke's theorem

Point: Green's theorem can be recast using either ① curl $\overset{\text{or}}{\vee}$ ② divergence. Theorem (divergence)

These two ways of recasting Green's theorem lead to two separate generalizations of Green's theorem.

⌐——— final

§16.6 Parametric Surface

Definition: A parametric surface is a function $\vec{s}(u,v) = (x(u,v), y(u,v), z(u,v))$ for some domain $D \subseteq \mathbb{R}^2$

Idea: This is a "space curve of dimension 2"

Ex. A sphere of radius $r > 0$ can be parametrized as:

$$S(\theta, \varphi) = \langle r\sin\varphi\cos\theta, \; r\sin\varphi\sin\theta, \; r\cos\varphi \rangle$$

↳ comes from spherical coordinates

on $D = [0, 2\pi] \times [0, \pi]$

Ex. The torus has parameterization

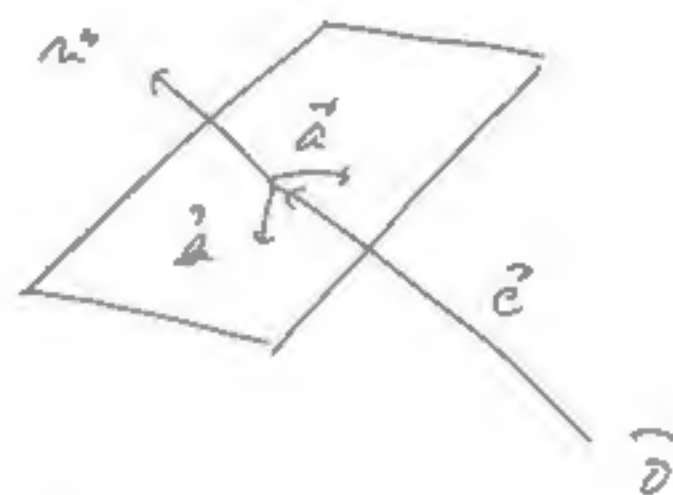$$\vec{S}(u,v) = \langle 2 + \cos v \cos u, \; (2 + \cos v)\sin u, \; \cos v \rangle$$

on $D = [0, 2\pi] \times [0, \pi]$

Ex. every plane $\overset{\text{in } \mathbb{R}^3}{v}$ can parameterized via

$$\vec{S}(u,v) = u\vec{a} + v\vec{b} + \vec{c} \quad \text{for suitable } \vec{a}, \vec{b}, \vec{c}$$

for $D = \mathbb{R}^2$

Idea: π is just determined by point (u,v)

in $\mathbb{R}^2$ via $\vec{a}, \vec{b}, \vec{c}$ & the eq above.

Ex. Compute a parameterization for the paraboloid $z = x^2 + 2y^2$

Note: there are many ways to parameterize this surface.

sol ① $\quad \vec{S}(x,y) = \langle x, y, x^2 + 2y^2 \rangle \quad D = \mathbb{R}^2$

sol ② $\quad \vec{S}(r,\theta) = \langle r\cos\theta, r\sin\theta, r^2(1 + \sin^2\theta) \rangle$

$D = [0, \infty) \times [0, 2\pi]$

sol ③ $\quad \vec{S}(r,\theta) = \langle \sqrt{2}\, r\cos\theta, r\sin\theta, 2r^2 \rangle$

Ex. Let $f(x)$ be a single-variable function. The surface determined by revolving f about the x-axis is parameterized by

$$\vec{S}(x,\theta) = \langle x, f(x)\cos\theta, f(x)\sin\theta \rangle$$

on $D = \text{dom}(f) \times [0, 2\pi]$

↳ sub-ex: Let $f(x) = x^3$

this surface has parameterization

$$S(x,\theta) = \langle x, x^3\cos\theta, x^3\sin\theta \rangle$$